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# Spatiotemporal chaos and effective stochastic dynamics for a near-integrable nonlinear system

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**Abstract**

We address (i) the onset of spatiotemporal chaos (STC) induced by the hyperbolic structure in a weakly perturbed nonlinear Schrödinger equation, and (ii) its effective stochastic dynamics (ESD). We obtain the following new results: (1) a very small number (*two* for our system) of linearly unstable modes is sufficient to trigger the onset of STC as characterized by an exponential decay in space of the mutual information; (2) the construction of the ESD needs only *temporal* chaos, in contrast to the requirement of full STC as usually believed. © 1999 Elsevier Science B.V.

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Chaos in deterministic finite-dimensional dynamical systems is a well-understood subject. However, most systems in physics and nature are described by partial differential equations (PDEs), which are infinite-dimensional dynamical systems. The simultaneous presence of temporal *and spatial* chaos in PDE is an essential component to fundamental limits for prediction in nature. Here by spatiotemporal chaos (STC) we mean *temporally chaotic dynamics together with long range spatial statistical independence.* Our focus will be the disappearance of coherence in space, i.e., the onset of spatial chaos, in a spatially extended dynamics whose time dynamics is chaotic as signified by positive Lyapunov exponents (sensitive dependence on initial conditions), or homoclinic tangles, etc. STC has posed a major conceptual and theoretical challenge in modern physics and mathematics [1,2]. There is a dearth of analytically tractable STC models [3]. Coupled map lattices have facilitated our understanding of many aspects of STC by drawing from the great reservoir of the knowledge about the original uncoupled maps [2]. In contrast, the understanding of STC for PDE has been limited and mainly concentrated on a few systems such as the complex Ginzburg–Landau equation and the Kuramoto–Sivashinsky (KS) equation [4,5]. On the other hand, spatially localized coherent structures induced by modulational instability have provided an important theoretical framework for studying complex dynamics in many fields, e.g., hydrodynamics, plasma physics, nonlinear optics, biophysics and condensed matter physics. In this regard, completely integrable PDEs possessing spatially coherent soliton solutions represent the most idealized framework. As is well known, a controlled weak breaking of integrability has

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led to surprisingly rich dynamics, while simultaneously allowing the utilization of analytical tools from soliton mathematics to depict underlying structures. This class of problems includes perturbed nonlinear Schrödinger equations (NLS) and perturbed sine-Gordon equations, where the existence and natures of *temporally* chaotic evolution of spatially localized structures have been firmly established and have been linked to linear instabilities in integrable soliton dynamics [6]. The underlying hyperbolic structures described by homoclinic orbits of the full PDEs are the sources of sensitivity which can engender chaotic responses in a perturbed dynamics and are the origin of the chaotic behavior observed in many systems, including conservative lattice dynamics [7]. Thus, a natural and important question arises, i.e., whether these hyperbolic structures can give rise to the phenomenon of STC [8]. In addressing this, we stress the importance of using mutual information (MI) in quantifying spatial chaos as it captures stronger statistical independence than two-point correlations (since a vanishing MI is a necessary and sufficient condition for statistical independence).

A second important issue in the statistical physics of these temporally and/or spatially chaotic systems is whether they admit effective stochastic models for long wavelength dynamics. Recently, this issue has received renewed interest, particularly in the connection between the Burgers–KPZ universality class and the hydrodynamic limit of the KS system [5,9]. We note that various extensions of the concepts of thermodynamics to nonequilibrium have been attempted for systems which exhibit turbulence or STC [10]. To understand the thermodynamics of NLS, a statistical mechanical formalism was extended to construct a Gibbs measure [11]. Here we will take an effective stochastic dynamics (ESD) approach, which is an extension to a dissipative dynamics of the Zwanzig–Mori projection formalism for a Hamiltonian system in thermal equilibrium [12]. A similar procedure has been used for the KS model in the STC regime, resulting in the noisy Burgers equation as the effective long wavelength dynamics [5]. We address an important question in this formalism, viz., what aspect of chaoticity is required to achieve a successful ESD construction: Is STC necessary or is temporal chaos sufficient to construct a coarse-grained dynamics?

In this Letter, we focus on the issue of STC in a weakly perturbed NLS and its ramification for the ESD. We study the relationship between the number of linearly unstable modes (LUM) and the onset of STC. Surprisingly, we find that (i) contrary to the implied belief that a large number of LUMs are required for the onset of STC [1,3–5,10] a very small number (namely, *two* for our system) of LUMs is sufficient to trigger the onset of STC, while the evolution of the system with only one LUM exhibits only temporal chaos; (ii) The coarse-grained dynamics for NLS requires only temporal chaos, in contrast to KS for which the STC is believed to play a significant role for the validity of ESD [5].

The dynamics we study is the driven, damped NLS

$$\mathrm{i}\psi_t + \psi_{xx} + 2|\psi|^2\psi = -\mathrm{i}\alpha\psi + \Gamma \mathrm{e}^{\mathrm{i}(\omega t+\gamma)}, \tag{1}$$

with periodic boundary conditions $\psi(x+L) = \psi(x)$. $L$ is the system length. $\omega$ and $\gamma$ are the driving frequency and phase, respectively. Near-integrability requires the damping $\alpha$ and the driving strength $\Gamma$ to be small. We note that for the one LUM case the following route to chaos has been verified numerically and analytically [6]. For a fixed small $\alpha$, increasing the bifurcation parameter $\Gamma$ of the driving, the long time dynamics undergoes a space-dependent quasi-periodic route to temporal chaos, changing from regular patterns in both space and time to regular spatial patterns that evolve chaotically in time. The solution consists of irregular jumps of a quasi-soliton by half system length ($\frac{1}{2}L$-jumps) [1]. The temporal chaos has been confirmed by a broad-band power spectrum, positive Lyapunov exponents, scattered Poincaré sections, fractional Lyapunov dimensions, etc. [6]. An example of this spatially regular, temporally chaotic evolution is shown in Fig. 1a, where we start with an initial condition which contains only one LUM. This kind of initial conditions can be easily obtained by studying the linearized stability of, say, a spatially independent solution: $\psi(x,t) = A\exp(2\mathrm{i}|A|^2 t)$. This solution has LUMs with the growth rate $\lambda = \pm k\sqrt{4|A|^2 - k^2}$ for $|k| < 2|A|$, $k = 2\pi m/L$. For a fixed value of $|A|$, one can control the number of LUMs by tuning the system length $L$. In the integrable limit of Eq. (1), i.e., $\alpha = \Gamma = 0$, these linear instabilities can be exponentiated

[1] If even symmetry were imposed, the quasi-soliton would locate only at two locations, the center and the edge of the system, thus, a center-edge jump [6].

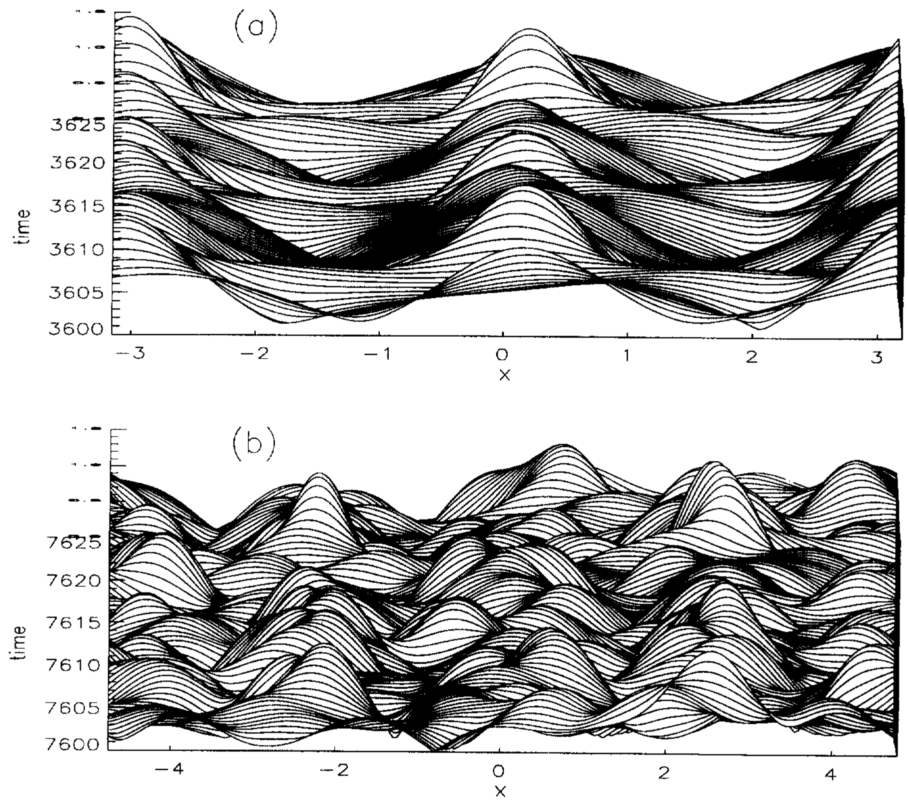


Fig. 1. Evolution of system (1) with $\alpha = 0.004$, $\Gamma = 0.144$, $\omega = 1$. The initial condition $\psi = A + \epsilon\exp(i2\pi x/L)$, $A = 0.8$, $\epsilon = 2\times10^{-5}$. Plotted here is $|\psi(x,t)|$. (a) Temporal chaos in the presence of one LUM, $L = 6.4$; (b) STC in the presence of two LUMs, $L = 9.6$.

via Bäcklund transformations to obtain global representations of the homoclinic orbits. These homoclinic orbits and their target tori have complicated spatiotemporal structures (e.g., the spatially independent solution above is one of the simplest examples of these targets). For the one LUM case, the perturbation induces homoclinic crossings as signified by the $\frac{1}{2}L$-jumps of one quasi-soliton, which constitutes, by translation invariance, essentially a class of one spatial structure, i.e., a quasi-soliton – hence, spatial correlation (as confirmed below). When more than one unstable mode is present, there is an increasingly large number of distinct classes of spatial excitations in forms of coexistence of many quasi-solitons or right and left traveling, spatially periodic or quasiperiodic nonlinear waves. A similar scenario occurs in studies of purely spatially chaotic, stationary waves [13]. Whereas, in our case the instabilities associated with homoclinic orbits provide needed sensitivity for chaotic wanderings among these spatial excitations under perturbations. The question is how many LUMs are needed to generate sufficiently many distinct spatial structures for the orbit to visit so as to reduce the spatial correlation, hence the onset of STC.

To characterize spatially chaotic behavior, we first use the spatial correlation function $C(x) = \langle\psi(x',t')\psi^*(x'+x,t')\rangle$. Average was taken over both space and time after sufficient long transients, e.g., $t > 2\times10^3$, were truncated. Fig. 2 shows the dependence of $C(x)$ on the system length. For $L = 6.4$, which corresponds to the one LUM case, clearly, the whole system is correlated (cf. Fig. 1a). However, for higher numbers of LUMs the system becomes increasingly decorrelated, signifying an onset of STC, as their correlation functions rapidly vanish. In Fig. 2 (inset), the correlation at the half system length as a function of $L$ shows a clear transition around $L_{th} = 2\pi/A$, above which the second LUM enters.

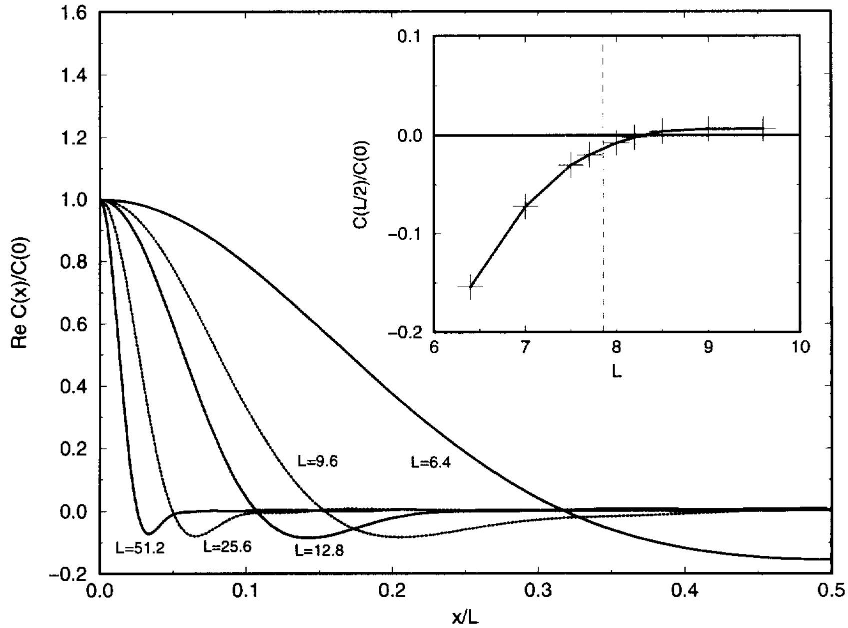


Fig. 2. Dependence of the correlation $C(x)$ on the system size $L$. Numerically, $\mathrm{Im}\,C(x)$ vanishes. Inset (see text): Transition of $C(\frac{1}{2}L)$ around $L_{\mathrm{th}} = 2\pi/A$ (dashed line). The symbol size indicates the numerical error estimate.

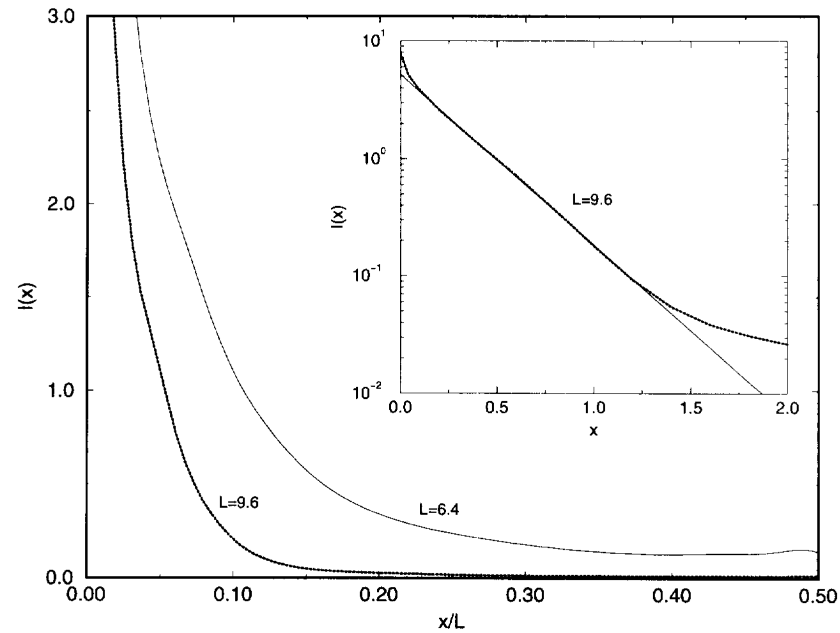


Fig. 3. Mutual information $\mathcal{I}(x)$. Fine line: one LUM; dotted line: two LUMs as also shown in the inset on the linear-log scale (the straight line is a fit to an exponential form).

The numerical value of $L$ above which $C(L/2) \sim 0$ differs from $L_{\text{th}}$ only by 4%. Fig. 1b shows a case with two LUMs, which displays drastically different spatial patterns from those in Fig. 1a. To further corroborate these results, we use an MI measure, which is given by $\mathcal{I}(x,y) = \int \mathrm{d}X\,\mathrm{d}Y\,P_{X,Y}\log[P_{X,Y}/(P_X P_Y)]$ between two spatial points $x$ and $y$. Here $X = \psi(x)$ and $Y = \psi(y)$. $P_{X,Y}$ is the joint probability density distribution between $X$ and $Y$. $P_X = \int P_{X,Y}\,\mathrm{d}Y$. By the translational symmetry of the system, $\mathcal{I}(x,y) = \mathcal{I}(x-y)$. As shown in Fig. 3, the MI for the one LUM case remains nonzero across the system while it vanishes *exponentially* for the two LUM case with a decay length $\xi \sim 0.30$. As solitons are phase-locked to the external driver, we argue that the driving frequency $\omega$ controls this decay length since the soliton's frequency determines its spatial width, i.e., the coherence length in space, while the driving strength $\Gamma$ changes $\xi$ as a higher order effect only.

Before turning to ESD for the system (1), we show in Fig. 4 that the system (1) reaches an equipartition of the power of the spatial Fourier modes for long wavelengths and the number of the Fourier modes in this "thermally" equilibrated subsystem is approximately the same as the number of LUMs. This equipartition has ramification for the statistical description of our system, e.g., enabling us to formulate a fluctuation–dissipation theorem for an ESD for the subsystem. Obviously, the observation poses an intriguing question for future investigations, i.e., what underpins the relation between linear instabilities and the equipartition of these modes. To construct the ESD for Eq. (1) we follow Zaleski [5] and for an arbitrary wavenumber $\Lambda$ and any $\tilde{\alpha}_k$, we can rewrite the Fourier transform of Eq. (1) in an equivalent form:

$$\mathrm{i}\dot{a}_k = (k^2 - \mathrm{i}\tilde{\alpha}_k)a_k - (2/L^2)\sum_{q,p}^{<} a_q a_p a^*_{p+q-k} + L\Gamma \mathrm{e}^{\mathrm{i}(\omega t+\gamma)}\delta_{k,0} + F_k(t), \qquad (2)$$

where $F_k(t) = -\mathrm{i}(\alpha - \tilde{\alpha}_k)a_k - 2/L^2 \sum' a_q a_p a^*_{q+p-k}$. $\sum^<$ denotes summation over all $|q|$, $|p|$, $|q+p-k| < \Lambda$ and $\sum'$, a summation in which at least one of $|q|$, $|p|$, $|q+p-k|$ is no less than $\Lambda$. Next, we view the chaos in the dynamics as a stochastic source for some selected modes, say, $|k| < \Lambda$, i.e., $\Lambda$ as a cutoff, and study how the dynamics of these modes is renormalized over some coarse-grained scales. In this setting, $F_k(t)$ can be naturally regarded as the external effective stochastic force and $\tilde{\alpha}_k$ as an effective correction to dissipation and/or dispersion (e.g., a $k$-dependent $\mathrm{Re}\,\tilde{\alpha}_k$ will represent an effective $k$-dependent damping while $\mathrm{Im}\,\tilde{\alpha}_k$ an effective dispersion). $\tilde{\alpha}_k$ is determined by requiring causality,

$$\langle F_k(t)a^*_k(t-s)\rangle_t = 0, \quad \text{for } s > 0, \qquad (3)$$

where $\langle\cdots\rangle_t$ is the time average over $t$, which leads to

$$\tilde{\alpha}_k = \alpha - (2\mathrm{i}/L^2)\sum{}' \langle a_q a_p a^*_{q+p-k} a^*_k\rangle(s) / \langle a_k a^*_k\rangle(s), \qquad (4)$$

where $\langle\cdots a^*_k\rangle(s) \equiv \langle\cdots(t)a^*_k(t-s)\rangle_t$. Relation (4) implies explicit $s$-dependence (we will denote this as $\alpha_k(s)$ below). The existence of an ESD with this type of memory function requires an $s$-independence – at least over some time scale (a coarse-grained time). Our numerical computation of the full dynamics (1) shows that Eq. (4) is indeed $s$-independent for the perturbed NLS even in the regime of *only temporal chaos*, without requiring STC. Fig. 5 illustrates a case in the presence of the temporal chaos alone. From the inset of Fig. 5, it is evident that there is a time-scale $\tau \sim 2\pi/\omega$ (determined by the external driver) after which the numerically computed $\alpha_k(s)$ is nearly constant over time scale $\tau < s < \tau'$, where $\tau'$ is the half lifetime for the slow decay of $\langle a_k(t)a^*_k(t+s)\rangle_t$, $k < \Lambda$. We observe that the regime of $\tau < s < \tau'$ involves many averages of chaotic $\frac{1}{2}L$-jumps of a quasi-soliton (homoclinic crossings). Our results on the effective $\mathrm{Re}\,\tilde{\alpha}_k$ show that it gives rise to dissipation for $k < \Lambda$, with $\mathrm{Re}\,\tilde{\alpha}_k \sim \alpha$ (within our numerical error). The renormalization factor $\mathrm{Im}\,\tilde{\alpha}_k$ is $\Lambda$-dependent, e.g., for $\Lambda = 4\pi/L$, $L = 6.4$, $\mathrm{Im}\,\tilde{\alpha}_k = -0.475$, $k = 2\pi/L$ as shown in Fig. 5 (inset), while, for $\Lambda = 10\pi/L$, $\mathrm{Im}\,\tilde{\alpha}_k$ has the form of $\beta_0 + \beta_2 k^2 + \beta_4 k^4$, leading to a modification of the Schrödinger dispersion $\omega = k^2$ to $\omega \sim (1+\beta_1)k^2 + \beta_2 k^4$, $\beta_1 \sim -0.02$ and $\beta_2 \sim 0.0061$. In addition to this constant $\alpha_k(s)$ test, another criterion for a successful construction is that the effective stochastic force should have no long-time correlation. This is satisfied in our case (Fig. 5): $G_k(s) \equiv \langle F_k(t)F^*_k(t+s)\rangle_t$ decays rapidly with a decay time $\sim \tau$, i.e., it can be regarded as no correlation over the coarse-grained timescale. The construction procedure

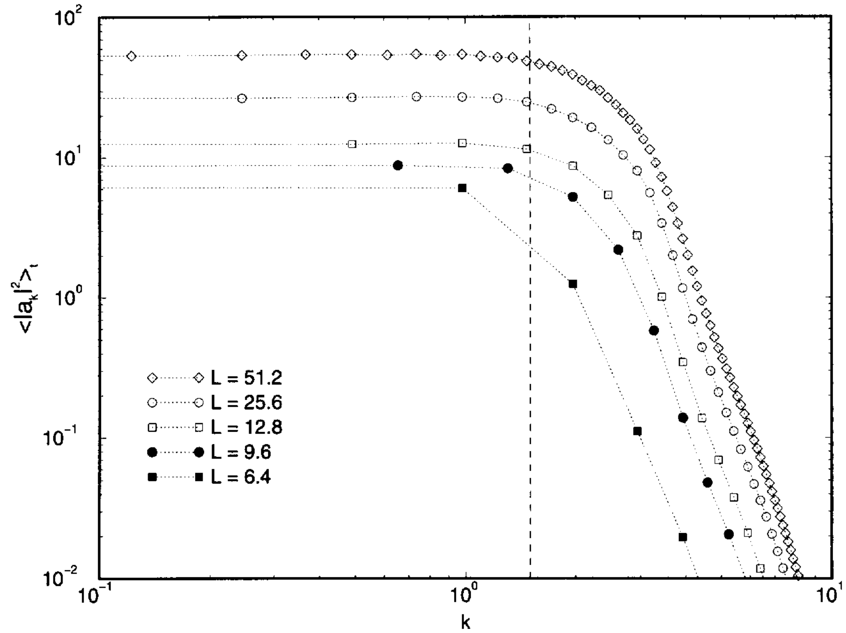


Fig. 4. Equipartition of the power of spatial Fourier modes in the long wavelength regime. The modes in the region left of the dashed vertical line are linearly unstable. Each point of any symbol labels one mode. The dotted lines between the symbols for guiding the eyes only.

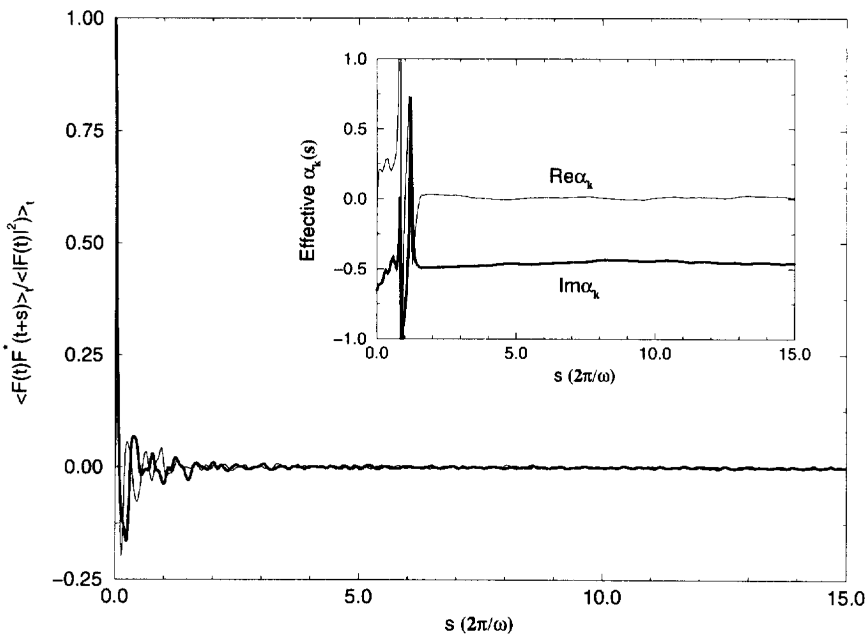


Fig. 5. Rapid decay of the force-force correlation $G_k(s)$ ($\mathrm{Re}\,G_k(s)$ (thick line) and $\mathrm{Im}\,G_k(s)$ (thin line)) and the effective $\alpha_k(s)$ (inset). The time unit is normalized by $2\pi/\omega$. $\tau'$ is considerably larger than the time-scale shown here (see text).

only demands the causality (3), which only involves correlations for the same $k$. $\langle F_k(t) a^*_{k'}(t-s)\rangle_t$ for $k \neq k'$ is left unconstrained. However, our numerical results indicate that, interestingly, for the $F_k(t)$ constructed above, $\langle F_k(t) a^*_{k'}(t-s)\rangle_t \approx 0$ for $\tau < s < \tau'$ also holds for $k \neq k'$. The fulfillment of causality in this general form is indicative of a deep self-consistency in the construction to validate the interpretation of $F_k(t)$ as an external stochastic forcing.

Chaotic behavior in systems with extended spatial domains is central to an understanding of nonlinear waves in nature; yet, today spatiotemporal chaos is the least understood of all chaotic behavior. Commonly held beliefs about spatiotemporal chaos include: (i) two-point correlation functions are usually adequate measures of the degree of spatial correlation; (ii) spatiotemporal chaos requires systems with very large (infinite) spatial extent, with many (infinite) instabilities; and (iii) the validity of "effective equations" to describe the behavior of the waves over long distances and times requires spatiotemporal chaos. The initial numerical experiments for the perturbed nonlinear Schrödinger equation, as described in this Letter, suggest that these three beliefs should be re-examined: (i) mutual information captures a spatial scale for the loss of coherence which can be missed by two-point correlation functions; (ii) for the NLS system, only two instabilities are needed for the onset of spatial decorrelation; and (iii) only temporal chaos seems needed for the validity of the effective dynamics.

While these initial studies are restricted to the NLS equation, they should apply to a large class of weakly perturbed, completely integrable nonlinear waves – provided the integrable system possesses instabilities. To see this, realize that each such instability is associated with a spatially localized wave, which under periodic boundary conditions (in contrast to Neumann boundary conditions), can reside anywhere in the periodic spatial domain. If two or more such localized waves are present, a measurement that locates one wave near a point $x$ gives no information about the location of the others. This uncertainty spatially decorrelates the system in the presence of two (or more) instabilities, and provides a general mechanism for the generation of spatiotemporal chaos as seen in our numerical study.

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